

COUNTING RATIONAL POINTS ON SMOOTH CUBIC SURFACES

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ABSTRACT. We prove that any smooth cubic surface defined over any number field satisfies the lower bound predicted by Manin's conjecture possibly after an extension of small degree.

1. INTRODUCTION

Let K be a number field. Assume $X \subset \mathbb{P}_K^3$ is a smooth cubic surface defined over K for which the set of rational points $X(K)$ is not empty. We are concerned with estimating the number of rational points of bounded height on X . Let $U \subset X$ be the Zariski-open set obtained by removing the lines contained in X , denote by H the exponential Weil height on $\mathbb{P}^3(K)$ and define for all $B \geq 1$ the counting function

$$N_{K,H}(U, B) := \#\{\mathbf{x} \in U(K) : H(\mathbf{x}) \leq B\}.$$

Manin's conjecture [FMT89] for smooth cubic surfaces states that

$$N_{K,H}(U, B) \sim cB(\log B)^{\rho_{X,K}-1}, \quad (1.1)$$

as $B \rightarrow \infty$, where $\rho_{X,K}$ denotes the rank of the Picard group of X over K and $c = c_{K,H,X}$ is a positive constant which was later interpreted by Peyre [Pey95].

There has been a wealth of results towards this conjecture but it has never been established for a single smooth cubic surface over any number field. There are proofs of Manin's conjecture for certain singular cubic surfaces over \mathbb{Q} , e.g. [BBD07], and other number fields [BT98, Fre13, DF15], but here we will only consider the smooth case. Heath-Brown [HB97], building upon the work of Wooley [Woo95], proved, using a fibration argument, that if X is a smooth cubic surface defined over \mathbb{Q} that contains 3 rational coplanar lines then $N_{\mathbb{Q},H}(U, B) \ll_{X,\epsilon} B^{\frac{4}{3}+\epsilon}$ holds for any $\epsilon > 0$. This result was subsequently extended to arbitrary number fields by Broberg [Bro01] and Browning and Swarbrick Jones [BSJ14]. Heath-Brown [HB98] revisited the subject by proving that a bound of the same order holds for all smooth cubic surfaces defined over \mathbb{Q} subject to a standard conjecture regarding the growth rate of the rank of elliptic curves. Using a generalization of Heath-Brown's determinant method, Salberger [Sal15] was able to prove unconditionally that one has $N_{\mathbb{Q},H}(U, B) \ll_{\epsilon} B^{\frac{12}{7}+\epsilon}$ for arbitrary smooth cubic surfaces defined over \mathbb{Q} and for all $\epsilon > 0$. Regarding lower bounds, the only available result is due to Slater and Swinnerton-Dyer [SSD98] who used a secant and tangent process to establish that $N_{\mathbb{Q},H}(U, B) \gg_X B(\log B)^{\rho_{X,\mathbb{Q}}-1}$ whenever X has 2 skew lines defined over \mathbb{Q} .

Our main result shows that for all smooth cubic surfaces over any number field L , the lower bound predicted by Manin's conjecture has the correct order of magnitude as soon as one passes to a sufficiently large extension of L . Some context for this

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type of result is provided by the formulation of Manin's conjecture in [BM90] and by the notion of potential density [Tsc06, §3].

Theorem 1.1. *Let X be any smooth cubic surface defined over any number field L . Then there exists an extension K_0 of L with $[K_0 : L] \leq 432$ such that for all number fields $K \supseteq K_0$ we have*

$$N_{K,H}(U, B) \gg B(\log B)^{\rho_{X,K}-1},$$

as $B \rightarrow \infty$, where the implicit constant depends at most on X and K .

We hope that the number 432 will serve as a useful benchmark for researchers in the area to compare the strength of other methods with in the future.

Our Theorem 1.1 is a consequence of Theorem 1.2 below, which furthermore provides an explicit description of K_0 . One can take K_0 to be any extension of L over which 2 skew lines of X are defined. The fact that there exists such a K_0 with $[K_0 : L] \leq 432 = 27 \cdot 16$ can be proved as follows. Since X contains exactly 27 lines, each of them is defined over an extension of degree at most 27. Since there are 16 complex lines skew to a line ℓ , a further extension of degree at most 16 ensures that a line skew to ℓ is defined.

Theorem 1.2. *Let X be a smooth cubic surface defined over any number field K such that X contains two skew lines defined over K . Then*

$$N_{K,H}(U, B) \gg B(\log B)^{\rho_{X,K}-1},$$

as $B \rightarrow \infty$, where the implicit constant depends only on K and X .

Theorem 1.2 is a generalization of Slater and Swinnerton-Dyer's result to arbitrary number fields. Our proof however is entirely different and more conceptual than the one of Slater and Swinnerton-Dyer. It relies on a conic bundle fibration of X and a number field version of the earlier work [Sof14] of the second author which allows us to count rational points on each conic individually.

This result is presented in Section 2, together with our main analytic tool, a variant of Wirsing's theorem. Theorem 1.2 will be proved in Sections 3 and 4.

Throughout this paper, all implied constants are allowed to depend on the cubic surface X and the underlying number field K , unless the contrary is explicitly stated.

2. PRELIMINARIES

We denote the degree of K by n , its discriminant by Δ_K , and its ring of integers by \mathcal{O}_K . We write Ω_∞ , Ω_0 and Ω_K for the sets of archimedean places, non-archimedean places, and all places of K , respectively. We will write h_K , R_K and μ_K for the class number, regulator and the group of roots of unity in K . Moreover, r_1 (resp. r_2) denotes the number of real (resp. complex) embeddings of K .

In the proof of Theorem 1.2, we only need to consider a special family of height functions on $\mathbb{P}^2(K)$. Let $\lambda = (\lambda_v)_{v \in \Omega_\infty} \in (0, \infty)^{\Omega_\infty}$. For every $v \in \Omega_K$, and $\mathbf{x} = (x, y, z) \in K_v^3$, let

$$\|\mathbf{x}\|_{\lambda,v} := \begin{cases} \max\{|x|_v, \lambda_v |y|_v, |z|_v\} & \text{if } v \in \Omega_\infty \\ \max\{|x|_v, |y|_v, |z|_v\} & \text{if } v \in \Omega_0. \end{cases} \quad (2.1)$$

Here, $|\cdot|_v$ is the unique absolute value on K_v extending the usual absolute value on \mathbb{Q}_p , if v lies over the place p of \mathbb{Q} . Let $n_v := [K_v : \mathbb{Q}_p]$. We consider heights on $\mathbb{P}^2(K)$ defined by

$$H_\lambda((x : y : z)) := \prod_{v \in \Omega_K} \|(x, y, z)\|_{\lambda,v}^{n_v}.$$

Let $C \subset \mathbb{P}_K^2$ be a nonsingular conic defined by a quadratic form $Q \in \mathcal{O}_K[x, y, z]$ and assume that $C(K) \neq \emptyset$, which implies that $C \cong \mathbb{P}_K^1$. The heights H_λ induce heights on $C(K)$ via the embedding $C \subset \mathbb{P}_K^2$. We are interested in estimating the quantity

$$N_{K, H_\lambda}(C, B) := \#\{\mathbf{x} \in C(K) : H_\lambda(\mathbf{x}) \leq B\}$$

when the underlying quadratic form has the special shape

$$Q = ax^2 + bxy + dxz + eyz + fz^2, \quad (2.2)$$

with $a, b, d, e, f \in \mathcal{O}_K$. It is a simple task to write down an explicit isomorphism between C and \mathbb{P}_K^1 . Let Π be the matrix

$$\Pi := \begin{pmatrix} b & e & 0 \\ -a & -d & -f \\ 0 & b & e \end{pmatrix},$$

and define

$$\mathbf{q}(u, v) := \Pi \cdot \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}. \quad (2.3)$$

Then the map $(u, v) \mapsto \mathbf{q}(u, v)$ induces an isomorphism $\mathbb{P}_K^1 \rightarrow C$. To measure the form Q and the height H_λ , we introduce quantities

$$\langle Q \rangle := \prod_{v \in \Omega_K} \max\{|a|_v, |b|_v, |d|_v, |e|_v, |f|_v\}^{n_v} \text{ and } M_\lambda := \prod_{v \in \Omega_\infty} \max\{1, \lambda_v^{-1}\}^{n_v}.$$

The following lemma is a number field version of [Sof14, Prop. 2.1], specialized to the heights H_λ and with a crude estimation of the error term.

Lemma 2.1. *There exist constants $\beta \in (0, 1/2)$ and $\gamma > 0$ which depend at most on K such that whenever $C \subset \mathbb{P}_K^2$ is a nonsingular conic defined by a quadratic form Q as in (2.2), and $\lambda \in (0, \infty)^{\Omega_\infty}$, then*

$$N_{K, H_\lambda}(C, B) = c_{K, \lambda, C} \cdot B + O\left(B^{1-\beta} (M_\lambda \langle Q \rangle)^\gamma\right),$$

for $B \geq 1$. The leading constant $c_{K, \lambda, C}$ is positive and is the one predicted by Peyre, and the implied constant in the error term depends only on K .

Of course, Manin's conjecture for conics with respect to arbitrary anticanonical height functions is already known [Pey95], so the novelty of Lemma 2.1 lies in the uniformity of the estimate in the coefficients of the underlying quadratic form.

The proof over \mathbb{Q} in [Sof14] is based on the parameterization of $C(K)$ via \mathbf{q} , which reduces the estimation of $N_{K, H_\lambda}(C, B)$ to a lattice point counting argument. The same reduction works over arbitrary number fields by considering primitive points with respect to a fixed set of representatives for the ideal classes of \mathcal{O}_K and suitably chosen fundamental domains for the action of the unit group. The resulting lattice point counting problem can then be solved using, for example, the main result from [BW14]. The special shape of the heights H_λ enters only here, to ensure definability in an o-minimal structure. Altogether, the passage from \mathbb{Q} to arbitrary number fields in the proof of Lemma 2.1 uses mostly arguments already given in [FP16], but is straightforward and much simpler. The proof provides explicit values $\beta = 1/(3n)$ and $\gamma = 4$, but we will not give further details here. For the purpose of proving Theorems 1.1 and 1.2 we do not need explicit values for β and γ since any polynomial saving in terms of B and any polynomial dependence on $\langle Q \rangle$ and M_λ in the error term suffices.

As usual, the constant $c_{K, \lambda, C}$ has an explicit expression of the form

$$c_{K, \lambda, C} = \frac{1}{2} \cdot \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{|\mu_K|} \cdot \frac{1}{|\Delta_K|} \cdot \prod_{v \in \Omega_K} \sigma_v, \quad (2.4)$$

with local densities σ_v given as follows. For $v \in \Omega_\infty$, we have

$$\sigma_v = \text{vol}\{(y_1, y_2) \in K_v^2 : \|\mathbf{q}(y_1, y_2)\|_{\lambda, v} \leq 1\} \cdot \begin{cases} 1 & \text{if } v \text{ is real,} \\ 4/\pi & \text{if } v \text{ is complex,} \end{cases} \quad (2.5)$$

where $\text{vol}(\cdot)$ denotes the usual Lebesgue measure on $K_v^2 \cong \mathbb{R}^{2n_v}$. For $v \in \Omega_0$ corresponding to a prime ideal \mathfrak{p} of \mathcal{O}_K , we have

$$\sigma_v = 1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2} + \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) \sum_{d \in \mathbb{N}} \frac{\rho_{\mathbf{q}}^*(\mathfrak{p}^d)}{\mathfrak{N}\mathfrak{p}^d}, \quad (2.6)$$

where, for any ideal \mathfrak{a} of \mathcal{O}_K , the function $\rho_{\mathbf{q}}^*(\mathfrak{a})$ is defined as

$$\#\{(\sigma, \tau) \in (\mathcal{O}_K/\mathfrak{a})^2 : \sigma\mathcal{O}_K + \tau\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K, \mathbf{q}(\sigma, \tau) \equiv \mathbf{0} \pmod{\mathfrak{a}}\}. \quad (2.7)$$

The following version of Wirsing's theorem is a straightforward generalization to number fields of [FI10, Theorem A.5]. Its proof is, mutatis mutandis, the same and therefore omitted.

Lemma 2.2. *Let g be a multiplicative function on nonzero ideals of \mathcal{O}_K that is supported on the set of squarefree ideals. Assume that we have*

$$\sum_{\mathfrak{N}\mathfrak{p} \leq x} g(\mathfrak{p}) \log(\mathfrak{N}\mathfrak{p}) = k \log x + O(1) \quad (2.8)$$

for all $x \geq 2$, with $k \geq -1/2$, where the sum runs over nonzero prime ideals \mathfrak{p} and the implied constant is allowed to depend at most on K and g . Assume, moreover, that

$$\prod_{w \leq \mathfrak{N}\mathfrak{p} < z} (1 + |g(\mathfrak{p})|) \ll \left(\frac{\log z}{\log w} \right)^{|k|} \quad (2.9)$$

holds for all $z > w \geq 2$ and that

$$\sum_{\mathfrak{p}} g(\mathfrak{p})^2 \log(\mathfrak{N}\mathfrak{p}) < \infty. \quad (2.10)$$

Then

$$\sum_{\mathfrak{N}\mathfrak{a} \leq x} g(\mathfrak{a}) = c_g (\log x)^k + O((\log x)^{|k|-1}),$$

with a positive constant c_g , where the implied constant depends at most on K and g .

3. COVERING THE CUBIC SURFACE WITH CONICS

Let K be a number field and $X \subset \mathbb{P}_K^3$ a smooth cubic surface containing two skew lines defined over K . The residual intersection of X with a plane containing the first line generically defines a smooth conic. The second line contained in X intersects each such plane in a point that necessarily lies in the residual conic, thus showing that it is isotropic over K .

The construction we have described does in fact yield a conic bundle morphism. A linear change of variables allows us to assume that the two skew K -lines are given by

$$x_0 = x_1 = 0 \quad \text{and} \quad x_2 = x_3 = 0,$$

whence the cubic form defining X has the shape

$$F = a(x_0, x_1)x_2^2 + d(x_0, x_1)x_2x_3 + f(x_0, x_1)x_3^2 + b(x_0, x_1)x_2 + e(x_0, x_1)x_3, \quad (3.1)$$

where $a, d, f \in \mathcal{O}_K[x_0, x_1]$ are linear forms and $b, e \in \mathcal{O}_K[x_0, x_1]$ are quadratic forms. Moreover, $F = x_0Q_0 - x_1Q_1$ with quadratic forms $Q_0, Q_1 \in \mathcal{O}_K[x_0, \dots, x_3]$,

and the nonsingularity of X implies that the morphism $\pi : X \rightarrow \mathbb{P}_K^1$ given on points by

$$(x_0 : x_1 : x_2 : x_3) \mapsto \begin{cases} (x_0 : x_1) & \text{if } (x_0, x_1) \neq (0, 0) \\ (Q_1(\mathbf{x}) : Q_0(\mathbf{x})) & \text{if } (Q_1(\mathbf{x}), Q_0(\mathbf{x})) \neq (0, 0) \end{cases}$$

is well-defined. The fibre $\pi^{-1}(s : t)$ is the residual conic in the plane $\Lambda_{(s:t)}$ defined by $tx_0 - sx_1 = 0$. For any choice of (s, t) , it is isomorphic to the plane conic $C_{(s,t)}$ defined by the quadratic form

$$Q_{(s,t)} := a(s,t)x^2 + d(s,t)xz + f(s,t)z^2 + b(s,t)xy + e(s,t)yz = 0 \quad (3.2)$$

via the isomorphism $\phi_{(s,t)} : \mathbb{P}_K^2 \rightarrow \Lambda_{(s:t)}$ given by $(x : y : z) \mapsto (sy : ty : x : z)$.

The discriminant locus of π is given by the quintic binary form

$$\Delta(s, t) := (ae^2 - bde + fb^2)(s, t),$$

which is separable owing to the nonsingularity of X (see [Sha94, II.6.4, Proposition 1]). This confirms that the resultant

$$W_0 := \text{Res}(b(s, t), e(s, t))$$

must be in $\mathcal{O}_K \setminus \{0\}$, since the square of any common divisor of $b(s, t)$ and $e(s, t)$ divides $\Delta(s, t)$.

Clearly, each $C_{(s,t)}$ contains the rational point $(0 : 1 : 0)$, which is tantamount to the conic bundle morphism having a section defined over K . By a standard argument (see, e.g., the paragraph following (1.6) in [BB11]), we have

$$\rho_{X,K} = 2 + r, \quad (3.3)$$

where $r = r(X, K)$ is the number of split singular fibres above closed points of \mathbb{P}_K^1 . Since the section meets exactly one component of every singular fibre, we see that all singular fibres are split. Consequently r equals the number of irreducible factors of $\Delta(s, t)$ in $K[s, t]$.

Using the conic fibration described above, we can reduce counting points on X to counting points on the fibres $\pi^{-1}(s : t)$ as follows:

$$N_{K,H}(U, B) = \sum_{(s:t) \in \mathbb{P}^1(K)} N_{K,H}(\pi^{-1}(s : t) \cap U, B).$$

Let \mathcal{G} be a fundamental domain for the action of \mathcal{O}_K^\times on $(K^\times)^2$ with the property that

$$\max\{|s|_v, |t|_v\} \ll \max\{|s|_w, |t|_w\} \ll \max\{|s|_v, |t|_v\} \quad (3.4)$$

holds for all $v, w \in \Omega_\infty$ and all $(s, t) \in \mathcal{G}$. We can construct such a fundamental domain using, for example, the method from [MV07, Section 4]. Define the set

$$\mathcal{B}(x) := \left\{ (s, t) \in \mathcal{O}_K^2 \cap \mathcal{G} : \begin{array}{l} H((s : t)) \leq x, \\ s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K, \\ \pi^{-1}(s : t) \text{ is nonsingular} \end{array} \right\}, \quad (3.5)$$

where $H((s : t))$ is the usual exponential Weil height on $\mathbb{P}^1(K)$. For the purpose of acquiring a lower bound it is sufficient to restrict the summation to points $(s : t)$ with representatives in $\mathcal{B}(B^\delta)$, with the value $\delta := \beta/(2(1 + \gamma))$. Then $N_{K,H}(U, B)$ is larger than

$$\sum_{(s,t) \in \mathcal{B}(B^\delta)} N_{K,H}(\pi^{-1}(s : t) \cap U, B) = \sum_{(s,t) \in \mathcal{B}(B^\delta)} N_{K,H}(\pi^{-1}(s : t), B) + O(B^{2\delta}),$$

by Schanuel's theorem, since every nonsingular conic contains at most 54 points lying on lines in X .

We use the isomorphism $\phi_{(s,t)}$ defined above to identify $\pi^{-1}(s : t)$ with the plane conic $C_{(s,t)}$ given by (3.2). The height H on $\pi^{-1}(s : t)$ is pulled back to the height

$H \circ \phi_{(s,t)} = H_{\lambda}$ on $C_{(s,t)}(K)$, with $\lambda_v := \max\{|s|_v, |t|_v\}$ for all $v \in \Omega_{\infty}$, making the succeeding equality apparent,

$$N_{K,H}(\pi^{-1}(s:t), B) = N_{K,H_{\lambda}}(C_{(s,t)}, B).$$

Clearly, $\langle Q_{(s,t)} \rangle \ll H((s:t))^2$, and due to (3.4) we have $M_{\lambda} \ll 1$. Lemma 2.1 therefore reveals that

$$N_{K,H}(\pi^{-1}(s:t), B) = c(s,t)B + O(B^{1-\beta}H((s:t))^{2\gamma}),$$

with an explicit formula for $c(s,t) := c_{K,\lambda,C_{(s,t)}}$ given below Lemma 2.1. Our choice of δ implies that

$$N_{K,H}(U, B) \gg B \mathfrak{S}(B^{\delta}) + O(B), \quad (3.6)$$

where

$$\mathfrak{S}(x) := \sum_{(s,t) \in \mathcal{B}(x)} c(s,t).$$

Our last undertaking is to show that the quantity $\mathfrak{S}(B^{\delta})$, the sum of the Peyre constants of the smooth conic fibres, provides the logarithmic factors appearing in Theorem 1.2.

4. THE PROOF OF THEOREM 1.2

For each place v of K , let $\sigma_v(s,t)$ be as in (2.5),(2.6), with the parameterizing functions $\mathbf{q} = \mathbf{q}_{(s,t)}$ defined as in (2.3) for the quadratic form $Q_{(s,t)}$, and the norms $\|\cdot\|_{\lambda,v}$ as in (2.1), with $\lambda_v = \max\{|s|_v, |t|_v\}$. Let ζ_K be the Dedekind zeta function of K and ϕ_K be Euler's totient function for nonzero ideals of \mathcal{O}_K . Moreover, for nonzero ideals \mathfrak{a} of \mathcal{O}_K , we define the multiplicative function

$$\phi_K^{\dagger}(\mathfrak{a}) := \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}}\right),$$

where the product extends over all prime ideals \mathfrak{p} dividing \mathfrak{a} . Clearly,

$$\frac{1}{\zeta_K(2)} \leq \frac{\phi_K^{\dagger}(\mathfrak{a})\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \leq 1$$

holds for all \mathfrak{a} .

Lemma 4.1 (The non-archimedean densities). *Let η be any positive constant and suppose $s, t \in \mathcal{O}_K$ fulfill $s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K$. Then we have*

$$\prod_{v \in \Omega_0} \sigma_v(s,t) \geq \frac{1}{\zeta_K(2)} \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq B^{\eta} \\ \mathfrak{a}|\Delta(s,t) \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\right)^2.$$

Proof. Let $\rho_{(s,t)}^*(\mathfrak{a}) := \rho_{\mathbf{q}_{(s,t)}}^*(\mathfrak{a})$ as in (2.7). Expanding the Euler product present in the lemma reveals its equality to

$$\frac{1}{\zeta_K(2)} \sum_{\mathfrak{a}} \frac{\rho_{(s,t)}^*(\mathfrak{a})}{\phi_K^{\dagger}(\mathfrak{a})\mathfrak{N}\mathfrak{a}} \geq \frac{1}{\zeta_K(2)} \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq B^{\eta} \\ \mathfrak{a}|\Delta(s,t) \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \frac{\rho_{(s,t)}^*(\mathfrak{a})}{\phi_K^{\dagger}(\mathfrak{a})\mathfrak{N}\mathfrak{a}}.$$

Let \mathfrak{a} be an ideal of \mathcal{O}_K with $\mathfrak{a} \mid \Delta(s,t)$ and $\mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K$. We proceed to show that $\rho_{(s,t)}^*(\mathfrak{a}) \geq \phi_K(\mathfrak{a})$. Since s^3W_0 and t^3W_0 can be expressed as \mathcal{O}_K -linear combinations of $b(s,t)$ and $e(s,t)$, we see that $b(s,t)\mathcal{O}_K + e(s,t)\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$. For every $\lambda \in \mathcal{O}_K/\mathfrak{a}$ with $\lambda\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$, let $u := \lambda e(s,t)$ and $v := -\lambda b(s,t)$. Then $\mathbf{q}_{(s,t)}(u,v) \equiv 0 \pmod{\mathfrak{a}}$, and thus $\rho_{(s,t)}^*(\mathfrak{a}) \geq \phi_K(\mathfrak{a})$. \square

Lemma 4.2 (The archimedean densities). *Suppose that s and t satisfy the assumption of Lemma 4.1. Then we have*

$$\prod_{v \in \Omega_\infty} \sigma_v(s, t) \gg \frac{1}{H((s : t))^2}.$$

Proof. The estimates

$$\begin{aligned} |b(s, t)|_v, |c(s, t)|_v &\ll \max\{|s|_v, |t|_v\}^2 \quad \text{and} \\ |a(s, t)|_v, |d(s, t)|_v, |f(s, t)|_v &\ll \max\{|s|_v, |t|_v\} \end{aligned}$$

hold for each place $v \in \Omega_\infty$. Hence, all $(y_1, y_2) \in K_v^2$ satisfying

$$|y_1|_v, |y_2|_v \ll \max\{|s|_v, |t|_v\}^{-1},$$

with a suitably small absolute implied constant, must fulfill $\|\mathbf{q}_{(s,t)}(y_1, y_2)\|_{\lambda,v} \leq 1$.

We therefore get that

$$\prod_{v \in \Omega_\infty} \sigma_v(s, t) \gg \prod_{v \in \Omega_\infty} \max\{|s|_v, |t|_v\}^{-2n_v} = H((s : t))^{-2}. \quad \square$$

By (2.4), Lemma 4.1 and Lemma 4.2, we obtain

$$\mathfrak{S}(B^\delta) \gg \sum_{(s,t) \in \mathcal{B}(B^\delta)} \frac{1}{H((s : t))^2} \sum_{\substack{\mathfrak{N} \mathfrak{a} \leq B^\eta \\ \mathfrak{a} | \Delta(s,t) \\ \mathfrak{a} + W_0 \mathcal{O}_K = \mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N} \mathfrak{a}} \right)^2. \quad (4.1)$$

We observe that, apart from the condition $(s, t) \in \mathcal{G}$ from (3.5), every expression involving (s, t) in the above formula is invariant under scalar multiplication of (s, t) by units in \mathcal{O}_K^\times . Hence, we may replace \mathcal{G} by another fundamental domain \mathcal{H} , which will enable us to continue our estimation of $\mathfrak{S}(x)$. We obtain a fundamental domain \mathcal{H}_0 for the action of \mathcal{O}_K^\times on $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ by making use of the embedding $K^\times \rightarrow (K \otimes_{\mathbb{Q}} \mathbb{R})^\times = \prod_{v \in \Omega_\infty} K_v^\times$ as well as the construction in [MV07, Section 4] for the trivial distance functions

$$N_v : K_v \rightarrow [0, \infty), \quad s \mapsto |s|_v.$$

The norm $N : K \rightarrow \mathbb{Q}$ extends to $K \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow \mathbb{R}$ in an obvious way. The sets $\mathcal{H}_0(T) := \{s \in \mathcal{H}_0 : |N(s)| \leq T\}$ clearly satisfy $\mathcal{H}_0(T) = T^{1/n} \mathcal{H}_0(1)$, and by [MV07, Lemma 3], the set $\mathcal{H}_0(1)$ is bounded with Lipschitz-parameterizable boundary. This enables us to perform lattice point counting arguments in the sets $\mathcal{H}_0(T)$ and their translates, via [MV07, Lemma 2] for example. We choose $\mathcal{H} := (\mathcal{H}_0 \cap K) \times K^\times \subset (K^\times)^2$ as our fundamental domain for the action of \mathcal{O}_K^\times on K^2 .

Partitioning into congruence classes modulo \mathfrak{a} yields

$$\mathfrak{S}(B^\delta) \gg \sum_{\substack{\mathfrak{N} \mathfrak{a} \leq B^\eta \\ \mathfrak{a} + W_0 \mathcal{O}_K = \mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N} \mathfrak{a}} \right)^2 \sum_{\substack{(\sigma, \tau) \bmod \mathfrak{a} \\ \sigma \mathcal{O}_K + \tau \mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K \\ \mathfrak{a} | \Delta(\sigma, \tau)}} G_{\sigma, \tau}(B^\delta, \mathfrak{a}), \quad (4.2)$$

where

$$G_{\sigma, \tau}(x, \mathfrak{a}) := \sum_{\substack{(s,t) \in (\mathcal{O}_K \cap \mathcal{H}_0) \times \mathcal{O}_K \\ s \mathcal{O}_K + t \mathcal{O}_K = \mathcal{O}_K \\ (s,t) \equiv (\sigma, \tau) \bmod \mathfrak{a} \\ H((s:t)) \leq x \\ C_{(s,t)} \text{ nonsingular}}} \frac{1}{H((s : t))^2}.$$

Lemma 4.3 (Lattice point counting). *Let $\sigma\mathcal{O}_K + \tau\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$. Then*

$$G_{\sigma,\tau}(x, \mathfrak{a}) \gg \frac{\log x}{\mathfrak{N}\mathfrak{a} \phi_K(\mathfrak{a}) \phi_K^\dagger(\mathfrak{a})} + O\left(x^{-\frac{1}{2n}} \log x\right).$$

Proof. The discriminant $\Delta(s, t)$ is a quintic form whence the conic $C_{(s,t)}$ is singular for (s, t) lying on one of at most 5 lines through the origin in K^2 . Hence, there exists a constant $0 < \alpha < 1$, depending only on F and K , such that $C_{(s,t)}$ is nonsingular whenever $s, t \neq 0$ and $|t|_v < \alpha |s|_v$ holds for all $v \in \Omega_\infty$. Observe that for such (s, t) with $s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K$ we have $H((s : t)) = |N(s)|$. This shows that

$$G_{\sigma,\tau}(x, \mathfrak{a}) \gg \sum_{\substack{(s,t) \in (\mathcal{O}_K \cap \mathcal{H}_0) \times \mathcal{O}_K \\ s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K \\ (s,t) \equiv (\sigma,\tau) \pmod{\mathfrak{a}} \\ x^{1/2} \leq |N(s)| \leq x \\ 0 < |t|_v < \alpha |s|_v \forall v \in \Omega_\infty}} |N(s)|^{-2} =: G(x), \text{ say.}$$

Using Möbius inversion to remove the coprimality condition, we see that

$$G(x) = \sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} + \mathfrak{a} = \mathcal{O}_K}} \mu_K(\mathfrak{d}) \sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ s \equiv \sigma \pmod{\mathfrak{a}} \\ x^{1/2} \leq |N(s)| \leq x}} |N(s)|^{-2} \sum_{\substack{t \in \mathfrak{d} \\ t \equiv \tau \pmod{\mathfrak{a}} \\ 0 < |t|_v < \alpha |s|_v \forall v \in \Omega_\infty}} 1.$$

The condition $\mathfrak{d} + \mathfrak{a} = \mathcal{O}_K$ comes from our hypothesis that $\sigma\mathcal{O}_K + \tau\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$. The sum over t is just counting ideal-lattice points in a translated “box”, and their number is well known to be

$$\frac{c_K \alpha^n |N(s)|}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})} + O\left(\left(\frac{\alpha^n |N(s)|}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})}\right)^{(n-1)/n} + 1\right),$$

with a positive constant c_K depending only on K (see, for example, the proof of [FP16, Lemma 7.1]). Hence,

$$\begin{aligned} G(x) &= \frac{c_K \alpha^n}{\mathfrak{N}\mathfrak{a}} \sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} + \mathfrak{a} = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}} \sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ s \equiv \sigma \pmod{\mathfrak{a}} \\ x^{1/2} \leq |N(s)| \leq x}} \frac{1}{|N(s)|} + O\left(\sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x}} \sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ x^{1/2} \leq |N(s)| \leq x}} \frac{1}{|N(s)|^2}\right) \\ &\quad + O\left(\frac{1}{\mathfrak{N}\mathfrak{a}^{(n-1)/n}} \sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x}} \frac{1}{\mathfrak{N}\mathfrak{d}^{(n-1)/n}} \sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ x^{1/2} \leq |N(s)| \leq x}} \frac{1}{|N(s)|^{1+1/n}}\right). \end{aligned}$$

The sums over s in the error terms are taken over principal ideals of \mathcal{O}_K contained in \mathfrak{d} . For any $a > 0$, we have

$$\sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ x^{1/2} \leq |N(s)| \leq x}} \frac{1}{|N(s)|^{1+a}} \ll \sum_{\substack{\mathfrak{b} \in [\mathfrak{d}^{-1}] \\ \mathfrak{N}\mathfrak{b} \geq x^{1/2} \mathfrak{N}\mathfrak{d}^{-1}}} \frac{1}{\mathfrak{N}(\mathfrak{b}\mathfrak{d})^{1+a}} \ll \frac{1}{\mathfrak{N}\mathfrak{d} x^{a/2}}.$$

This shows that both error terms in the above expression for $G(x)$ are $\ll x^{-1/(2n)} \log x$. Using the nice properties of our fundamental domain \mathcal{H}_0 and [MV07, Lemma 2], we see that

$$\#\{s \in \mathfrak{d} \cap \mathcal{H}_0 : s \equiv \sigma \pmod{\mathfrak{a}}, |N(s)| \leq x\} = \frac{c'_K x}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})} + O\left(\left(\frac{x}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})}\right)^{(n-1)/n} + 1\right),$$

with a positive constant c'_K depending only on K . Together with the Abel sum formula we are thus provided with the asymptotic formula

$$\sum_{\substack{s \in \mathfrak{d} \cap \mathcal{H}_0 \\ s \equiv \sigma \pmod{\mathfrak{a}} \\ x^{1/2} \leq |N(s)| \leq x}} \frac{1}{|N(s)|} = \frac{c'_K}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})} \log(x) + O\left(x^{-1/(2n)}\right),$$

from which it is immediately apparent that

$$G(x) \gg \frac{\log x}{\mathfrak{N}\mathfrak{a}^2} \sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} + \mathfrak{a} = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}^2} + O\left(x^{-1/(2n)} \log x\right).$$

Finally, the obvious estimate

$$\sum_{\substack{\mathfrak{N}\mathfrak{d} \leq x \\ \mathfrak{d} + \mathfrak{a} = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\mathfrak{d}^2} = \frac{\mathfrak{N}\mathfrak{a}}{\zeta_K(2)\phi_K(\mathfrak{a})\phi_K^\dagger(\mathfrak{a})} + O\left(\frac{1}{x}\right)$$

allows us to complete the proof of the lemma. \square

For any binary form $g \in \mathcal{O}_K[u, v]$, we define the multiplicative function $\varrho_g^*(\mathfrak{a})$ on non-zero ideals of \mathcal{O}_K by

$$\sharp\{(\sigma, \tau) \in (\mathcal{O}_K/\mathfrak{a})^2, \sigma\mathcal{O}_K + \tau\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K, g(\sigma, \tau) \equiv 0 \pmod{\mathfrak{a}}\}$$

and note that its value is trivially bounded by $\mathfrak{N}\mathfrak{a}^2$. From the estimate (4.2) with $\eta := \delta/(7n)$ and Lemma 4.3, we obtain

$$\mathfrak{S}(B^\delta) \gg \sum_{\substack{\mathfrak{N}\mathfrak{a} \leq B^{\delta/(7n)} \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \frac{\varrho_\Delta^*(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^2} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\right)^2 + O(1). \quad (4.3)$$

The following lemma is proved via an application of Wirsing's theorem and its validity implies that of Theorem 1.2.

Lemma 4.4. *For $x \geq 1$,*

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \frac{\varrho_\Delta^*(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^2} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}\right)^2 \gg (\log x)^r.$$

Proof. The form Δ factors as $a\Delta(s, t) = \prod_{i=1}^r \Delta_i(s, t)$ over K for an appropriate value of $a = a(K, F) \in \mathcal{O}_K$ and irreducible forms $\Delta_i \in \mathcal{O}_K[s, t]$. For $1 \leq i \leq r$ with $\Delta_i(1, 0) \neq 0$, let $\delta_i(x) := \Delta_i(x, 1) \in \mathcal{O}_K[x]$. We moreover define for any polynomial $g \in \mathcal{O}_K[x]$ and any ideal \mathfrak{a} of \mathcal{O}_K ,

$$\tau_g(\mathfrak{a}) := \sharp\{s \in \mathcal{O}_K/\mathfrak{a} : g(s) \equiv 0 \pmod{\mathfrak{a}}\},$$

and we subsequently let

$$\tau_i(\mathfrak{a}) := \begin{cases} \tau_{\delta_i}(\mathfrak{a}) & \text{if } \Delta_i(1, 0) \neq 0, \\ \tau_x(\mathfrak{a}) = 1 & \text{if } \Delta_i(1, 0) = 0, \end{cases} \text{ and } a_i := \begin{cases} \Delta_i(1, 0) & \text{if } \Delta_i(1, 0) \neq 0, \\ 1 & \text{if } \Delta_i(1, 0) = 0. \end{cases}$$

The asymptotic relationships

$$\sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\tau_i(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}} = \log \log x + O(1) \quad \text{and} \quad \sum_{\mathfrak{N}\mathfrak{p} \leq x} \frac{\tau_i(\mathfrak{p}) \log(\mathfrak{N}\mathfrak{p})}{\mathfrak{N}\mathfrak{p}} = \log x + O(1) \quad (4.4)$$

follow from Landau's prime ideal theorem applied to $K(\theta_i)$, where θ_i is a root of the irreducible polynomial δ_i . Since Δ is separable, all resultants $\text{Res}(\Delta_i, \Delta_j)$ are nonzero. Whence, upon introducing

$$W = W_F := aW_0 \prod_{i \neq j} \text{Res}(\Delta_i, \Delta_j) \prod_{i=1}^r a_i \in \mathcal{O}_K \setminus \{0\},$$

the equality

$$\varrho_{\Delta}^*(\mathfrak{p}) = (\mathfrak{N}\mathfrak{p} - 1) \sum_{i=1}^r \tau_i(\mathfrak{p})$$

is rendered valid for each nonzero prime ideal \mathfrak{p} of \mathcal{O}_K , coprime to W . This fact, along with (4.4), reveals that the multiplicative function defined by

$$g(\mathfrak{a}) := \begin{cases} \varrho_{\Delta}^*(\mathfrak{a}) \phi_K(\mathfrak{a})^2 \mathfrak{N}\mathfrak{a}^{-4} & \text{if } \mathfrak{a} + W\mathcal{O}_K = \mathcal{O}_K \text{ and } \mathfrak{a} \text{ squarefree,} \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the assumptions of Lemma 2.2 with $k = r$. We therefore get that there exists $c_g > 0$ such that

$$\sum_{\substack{\mathfrak{N}\mathfrak{a} \leq x \\ \mathfrak{a} + W\mathcal{O}_K = \mathcal{O}_K \\ \mathfrak{a} \text{ squarefree}}} \frac{\varrho_{\Delta}^*(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^2} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \right)^2 = c_g (\log x)^r + O((\log x)^{r-1}),$$

an estimate which concludes our proof. \square

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